

$$(6) \quad C_{rst} = (-2)^{r+s+t} \sum_n 2^{-2n} \frac{n!}{(n-r)!(n-s)!(n-t)!(r+s+t-2n)!},$$

where in the summation all factorials have non-negative arguments.

If we transpose the right hand side of (5) and equate coefficients of  $u_1^r u_2^s u_3^t$ , we have

$$(7) \quad C_{rst} = C_{r,s-1,t-1} + C_{r-1,s,t-1} + C_{r-1,s-1,t} + 2C_{r-1,s-1,t-1},$$

which is the recurrence formula with simplest coefficients, especially as the right hand side can be treated as the sum of five terms. Even though all suffixes vary, it probably provides the quickest way of computing all values of  $C_{rst}$  up to a given set of  $r, s, t$ . For some machines at least, it may well give the quickest way for calculating a given  $C_{rst}$ , and it provides an easy method for desk-machine computation when  $r, s$  and  $t$  are small. For computations by other methods it provides a simple check. Other checks may be obtained by giving  $u_1$  special values in (5) and equating coefficients of  $u_2^s u_3^t$ . Putting  $u_1 = -1$ , we get

$$(8) \quad \sum_r C_{rst} = 1,$$

given by Gillis and Weiss; putting  $u_1 = -\frac{1}{2}$ , we get

$$(9) \quad \sum_r 2^{s+t-r} C_{rst} = \binom{s+t}{s}.$$

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1. J. GILLIS & G. WEISS, "Products of Laguerre polynomials," *Math. Comp. (MTAC)*, v. 14, 1960, p. 60-63.
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## The Evaluation of Certain Probability Integrals

By Irwin Greenberg

A problem which often arises in the field of probability and statistics is the following:

Assume that there are  $n$  independent stochastic processes and that the  $k$ th process has an output distribution  $f_k(x_k)$ . The probability that the  $j$ th process yields a higher output than any of the others is

$$(1) \quad P_j = \int_{-\infty}^{\infty} f_j(x_j) \prod_{\substack{k=1 \\ k \neq j}}^n \int_{-\infty}^{x_j} f_k(x_k) dx_k dx_j.$$

In certain special cases, equation (1) is easily integrated; for example, if the various  $f_k(x_k)$  are all uniform or exponential distributions.

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In certain other cases, equation (1) can be put into a form which is readily evaluated by hand computation, using well known statistical reference tables. This paper will illustrate one of these cases; for  $n = 2$  or  $3$  and the  $x_k$  having a normal distribution.

Under the assumption of normality,

$$(2) \quad f_k(x_k) = (\sqrt{2\pi} \sigma_k)^{-1} \exp \left[ -\frac{1}{2} \left( \frac{x_k - \mu_k}{\sigma_k} \right)^2 \right]; \quad k = 1, 2, 3$$

where  $\mu_k$  and  $\sigma_k$  are the mean and standard deviation, respectively. Letting

$$(3) \quad X_k = \frac{x_k - \mu_1}{\sigma_1},$$

equation (2) becomes

$$(4) \quad f_k(X_k) = (\sqrt{2\pi} \sigma_k')^{-1} \exp \left[ -\frac{1}{2} \left( \frac{X_k - \mu_k'}{\sigma_k'} \right)^2 \right]$$

where

$$(5) \quad \mu_k' = \frac{\mu_k - \mu_1}{\sigma_1}; \quad \sigma_k' = \frac{\sigma_k}{\sigma_1}.$$

The probability that the output of process 1 exceeds the output of process 2 is

$$(6) \quad \begin{aligned} P_1 &= (2\pi\sigma_2')^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{X_1} \exp \left[ -\frac{1}{2} X_1^2 - \frac{1}{2} \left( \frac{X_2 - \mu_2'}{\sigma_2'} \right)^2 \right] dX_2 dX_1 \\ &= (\sqrt{2\pi})^{-1} \int_{-\infty}^{M_2} \exp \left( -\frac{t^2}{2} \right) dt, \end{aligned}$$

where

$$(7) \quad M_2 = -\mu_2' / \sqrt{(\sigma_2')^2 + 1}.$$

The simplification of equation (6) is obtained by expanding the exponent, grouping terms, completing the square in  $X_1$  and reversing the order of integration after making the transformations:

$$(8) \quad X_1 = X_1'; \quad X_2 = X_1' + X_2' + \mu_2'.$$

Equation (6) is the cumulative normal function and is tabulated in most texts on statistics. In a similar manner, it can be shown that the probability that the output of process 1 exceeds the output of process 2 and of process 3 is

$$(9) \quad \begin{aligned} P_1 &= [(2\pi)^{3/2} \sigma_2' \sigma_3']^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{X_1} \int_{-\infty}^{X_1} \exp \left[ -\frac{1}{2} X_1^2 - \frac{1}{2} \left( \frac{X_2 - \mu_2'}{\sigma_2'} \right)^2 \right. \\ &\quad \left. - \frac{1}{2} \left( \frac{X_3 - \mu_3'}{\sigma_3'} \right)^2 \right] dX_3 dX_2 dX_1 \\ &= \left[ 2\pi \sqrt{1 - r^2} \right]^{-1} \int_{-\infty}^{M_2} \int_{-\infty}^{M_3} \exp \left[ -\frac{1}{2(1 - r^2)} (t_2^2 - 2rt_2 t_3 \right. \\ &\quad \left. + t_3^2) \right] dt_3 dt_2, \end{aligned}$$

where

$$(10) \quad \begin{cases} M_2 = -\mu_2' / \sqrt{(\sigma_2')^2 + 1} \\ M_3 = -\mu_3' / \sqrt{(\sigma_3')^2 + 1} \\ \frac{1}{r} = \sqrt{(\sigma_3')^2 + 1} \sqrt{(\sigma_2')^2 + 1}. \end{cases}$$

Equation (10) gives the volume under the bivariate normal probability surface with correlation coefficient  $r$ . These volumes are tabulated in [1].

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1. NBS Applied Mathematics Series, No. 50, *Tables of the Bivariate Normal Distribution Function and Related Functions*, U. S. Government Printing Office, Washington, D. C. 1959.

## The Congruence $2^{p-1} \equiv 1 \pmod{p^2}$ for $p < 100,000$

By Sidney Kravitz

Fröberg has previously announced [1] the computation of the Fermat remainders corresponding to all odd primes less than 50,000. His results show that  $p = 1093$  and  $p = 3511$  are the only solutions of the congruence  $2^{p-1} \equiv 1 \pmod{p^2}$  in that range.

The residues of  $2^{p-1} \pmod{p^2}$  have been computed for  $50,000 < p < 100,000$  on an IBM 650 system at Picatinny Arsenal. No residue congruent to 1 was found corresponding to a prime in this range.

A copy of the table of residues has been deposited in the Unpublished Mathematical Tables file.

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1. C. E. FRÖBERG, "Some Computations of Wilson and Fermat Remainders," *MTAC*, v. 12, 1958, p. 281.

**Editorial Note:** Reference should also be made to:

1. W. MEISSNER, "Über die Teilbarkeit von  $2^p - 2$  durch das Quadrat der Primzahl  $p = 1093$ ," *Akad. d. Wiss, Berlin, Sitzungsab.*, v. 35, 1913, p. 663-667

2. N. G. W. H. BEEGER, "On a new case of the congruence  $2^{p-1} \equiv 1 \pmod{p^2}$ ," *Messenger Math.*, v. 51, 1922, p. 149-150.

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